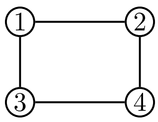
Exercises Discussion Session 1

Toric Varieties, Exponential Families, Examples of graphical models.

(1) Let (X_1, X_2, X_3) be a vector of binary random variables. Consider the following two graphs.

Write a polynomial parametrization of the model $\mathcal{M}(G_1)$ and write down the matrix A that defines the monomial parametrization of the model $\mathcal{M}(G_2)$. Find the implicit equations that define these models either by hand or by using a computer algebra software like Macaulay2. How do the two polynomial parametrizations compare?

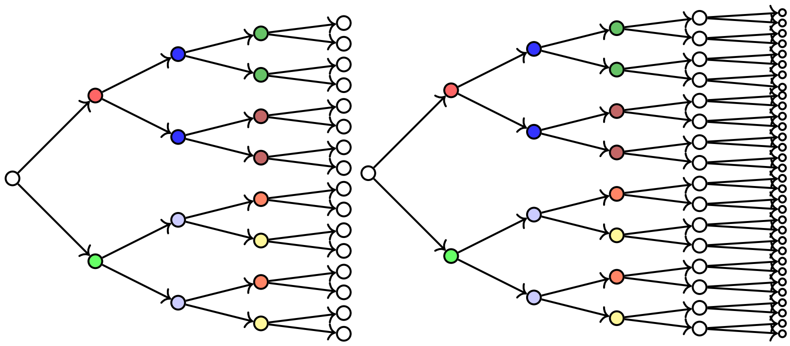
(2) Write down the matrix A for the undirected graphical model of the graph shown below and find its defining equations either by hand or using a computer algebra software like Macaulay2. Here the nodes of the graph represent discrete binary random variables.



(3) Prove that the family of pdfs $\{f(x|\mu, \sigma^2) : (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}, f \text{ is a normal density}\}$ is an exponential family.

(4) Why do is it that in Algebraic Statistics Δ°_{r-1} is identified with $\mathbb{P}^{r-1}_{\mathbb{C}}$

(5) Consider the following two staged trees $\mathcal{T}, \mathcal{T}'$. Prove that the tree on the left is balanced. Now consider $\mathcal{T}'$ on the left. What are all possible choices of stages you can make in the second to last level in such a way that $\mathcal{T}'$ is also balanced.



(4) Prove Proposition 1.3 in the slides, or read the proof from Sullivant (2018) Proposition 6.2.4., or ask a friend.

Extra remarks

- For a quick intro to ideals, polynomials and varieties check out Kahle u. a. (2018) available at the arxiv https://arxiv.org/pdf/1705.07411.pdf.
- The computer algebra package Macaulay2 is great to find implicit descriptions of varietes/statistical models. You can try it out here https://www.unimelb-macaulay2.cloud.edu.au/#home

```
-- Ring for the distributions
R=QQ[p00,p01,p10,p11]
-- Ring for the parameters
S=QQ[s0,s1,t0,t1]
-- A matrix for the model
A=matrix{{1,1,0,0},
         {0,0,1,1},
 {1,0,1,0},
 {0,1,0,1}}
needsPackage("FourTiTwo")
toricMarkov(A,R)
---
-- Homogeneous version
restart
SS=QQ[z,s0,s1,t0,t1]
-- sum to one condition ideal
sto1=ideal(s0+s1-1,t0+t1-1)
-- quotient ring
S=SS/sto1
R=QQ[p00,p01,p10,p11]
mons=z*{s0*t0,s0*t1,s1*t0,s1*t1}
phiG=map(S,R,mons)
I=ker(phiG)
```