Better bounds than BKK

MathCoRe Research Project of Andreas Kretschmer

Funded Fellow since September 2020 Advisors: Benjamin Nill, Thomas Kahle Mentor: Eliana Duarte

Summary

Given n Laurent polynomials $f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ in n variables, what is the number, counting multiplicities, of their isolated common zeros in $(\mathbb{C}^*)^n$? The general answer is provided by the Bernstein–Khovanskii–Kushnirenko Theorem (BKK), stating that the mixed volume $\mathrm{MV}(N(f_1), \dots, N(f_n))$ of the Newton polytopes $N(f_i)$ gives an upper bound which is generically an equality. Bernstein also gives the precise condition for when equality is achieved. If this condition is violated, however, little is known about better bounds. For certain types of polynomial systems we propose equivalent lifted systems with more variables and strictly lower mixed volume.

Background and examples

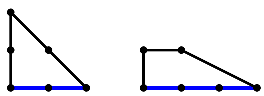
Given a convex polytope $P \subseteq \mathbb{R}^n$ and a vector $u \in \mathbb{R}^n \setminus \{0\}$, the subset P^u of P where the functional $\langle u, - \rangle$ is minimized is a face of P. For a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the *facial subpolynomial* f^u is the sum of the terms of f whose monomials lie in the face $N(f)^u$ of the Newton polytope $N(f)$. The following is the fundamental result on which this project builds.

Theorem 1 (BKK [1],[2]). *Let $f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be Laurent polynomials. The number of isolated solutions to the system $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ is bounded by $\mathrm{MV}(N(f_1), \dots, N(f_n))$, counting multiplicities, with equality if and only if for every $u \in \mathbb{R}^n \setminus \{0\}$ the facial subsystem $f_1^u = \dots = f_n^u = 0$ has no solution in $(\mathbb{C}^*)^n$.*

Example 2. Consider the size $n = 2$ polynomial system

$$0 = 1 + 2x + x^2 - xy + y^2,$$
$$0 = 1 + x - x^2 - x^3 + 2xy + y.$$

What is its number of solutions in $(\mathbb{C}^*)^2$? The Newton polytopes of these two polynomials are



and their mixed volume is 6. By Theorem 1, its exact number of solutions, however, is ≤ 5. The lifting

$$0 = z - xy + y^2,$$
$$0 = z(1 - x) + 2xy + y,$$
$$0 = z - (x + 1)^2,$$

has mixed volume 4, and using Theorem 1 again, it can be seen that this is the true number of solutions in $(\mathbb{C}^*)^3$, hence also the true number of torus solutions of the original size 2 system.

Relation to Complexity Reduction

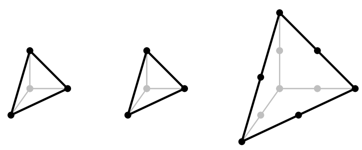
From the point of view of polyhedral homotopy continuation, passing from a non-generic polynomial system to a lifting deletes paths that tend to infinity along the homotopy. This indicates how introducing new variables decreases computational runtime in solving a polynomial system.

First results and outlook

For non-generic polynomial systems one can consider substitutions of the form $y_i = p_i(x_1, \dots, x_n)$ with Laurent polynomials p_i. A *lifting* of a square polynomial system is any such substitution together with the new equations $0 = y_i - p_i(x_1, \dots, x_n)$. A lifting is called *valid* if the resulting system has strictly lower mixed volume.

Proposition 3. *Every non-generic size 2 polynomial system has a valid lifting with one new variable.*

A *dense* polynomial system is one for which every Newton polytope is a dilation of the standard simplex:



Conjecture 4. *Every dense non-generic size n polynomial system has a valid lifting with $n - 1$ new variables.*

Proposition 5. *Assume the condition of Theorem 1 is violated in direction u and $f_i^u = \lambda f_j^u$ for some $i \neq j$ and $\lambda \neq 0$. Then the substitution $y = f_i^u$ gives a valid lifting.*

The proofs use a recent characterization of strict monotonicity of the mixed volume due to Bihan and Soprunov [3].

Question 6. *What are conditions to ensure that the lifted systems in the above cases are generic? Are there quantitative improvements of Theorem 1? Do there exist non-generic polynomial systems without any valid lifting?*

Publications and References

[1] David N. Bernštein. The number of roots of a system of equations (in Russian). Funk. Anal. Priloz. 9 (1975) 1-4. *English translation in Functional Anal. Appl*, 9:183–185, 1975.

[2] Askold G. Khovanskii. Newton polyhedron, Hilbert polynomial, and sums of finite sets. *Functional Analysis and Its Applications*, 26(4):276–281, 1992.

[3] Frédéric Bihan and Ivan Soprunov. Criteria for strict monotonicity of the mixed volume of convex polytopes. *Advances in Geometry*, 19(4):527–540, 2019.

Cooperations Tobias Boege, Christopher Borger, Thomas Kahle, Benjamin Nill, Frank Röttger, Alexander Ryzhkov, Sebastian Sager, Jonas Schulze