Implicit equations of tensor product surfaces via virtual projective resolutions

MathCoRe Research Project of Eliana Duarte

Funded postdoc since 2017

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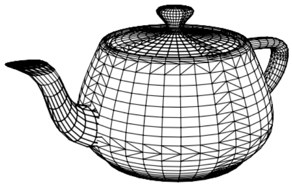
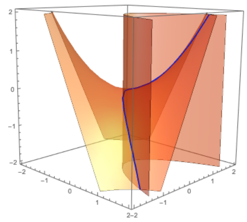
Summary

An algebraic surface in space can be described parametrically or implicitly. Parameterized surfaces are the building blocks of 3D objects in geometric modeling. To do geometric operations with these objects it is essential to have their implicit representations. We consider the problem of finding implicit equations of tensor product surfaces with basepoints. We develop the theory of residual resultants in $\mathbb{P}^1 \times \mathbb{P}^1$ to formulate a new algorithm that computes the implicit equations in this case.





Main Results [1]

1. Generalization of residual resultants in $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$.
2. New algorithm to compute the residual resultant using virtual resolutions and a variation of it to compute the implicit equation of tensor product surfaces with basepoints.

Virtual resolutions reduce the complexity

Free resolutions:

- An algebraic tool to study algebraic varieties in projective space $\mathbb{P}^n$, Hilbert(1890).

Virtual resolutions

- Appeared in the algebraic geometry literature in the last thirty years, their formalism was established in 2016 by [2].
- Less restrictive version of free resolutions, simpler and better to capture the geometry of varieties in multiprojective spaces, e.g. $\mathbb{P}^1 \times \mathbb{P}^1$.

Using resolutions to find implicit equations

- Free resolutions play a leading role in finding implicit equations of parameterized curves and surfaces [3].
- Virtual resolutions are also useful in implicitization, they are shorter and more manageable to find the needed value of the **regularity** to be able to perform the implicit equation computation. (See detailed example)

Getting started

A **parameterized tensor product surface** is the closure of the image of a polynomial map $\lambda : \mathbb{R}^2 \to \mathbb{R}^3$.
Consider the homogenization $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, where

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &= \text{compactification of } \mathbb{R}^2 \text{ into the product of two} \\ &\quad\ \text{projective lines, with coordinates } (s:t) \times (u:v), \\ \mathbb{P}^3 &= \text{compactification of } \mathbb{R}^3 \text{ into projective space,} \\ &\quad\ \text{with coordinates } (X:Y:Z:1). \end{aligned}$$

The surface has **basepoints** if the polynomials that define λ have nonempty vanishing in $\mathbb{P}^1 \times \mathbb{P}^1$.

The **residual resultant** of the system of polynomial equations $\{p_0 = p_1 = p_2 = 0\}$ is a polynomial $\mathrm{Res}_G(p_0, p_1, p_2)$ on the coefficients of the system that vanishes if and only if the system has a solution outside the zero set of another prescribed system of polynomial equations G.

Detailed example of implicitization

Four polynomials f_0, f_1, f_2, f_3 in the variables s, t and u, v define the surface,

$$\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, (s:t)\times(u,v) \mapsto \quad (s^2v \ :\ stv \ :\ stu \ :\ t^2u).$$

$G =$ system of equations defining the two basepoints of λ.

Algorithm: Compute $\mathrm{Res}_G(f_3X - f_0, f_3Y - f_1, f_3Z - f_2)$ over the polynomial ring T in the variables s, t, u, v, X, Y, Z.

Step 1: The regularity is $(2, 0)$.

$$\text{Virtual resolution}: 0 \leftarrow T \leftarrow T^6 \leftarrow T^8 \leftarrow T^3 \leftarrow 0$$

$$\text{Free resolution}: 0 \leftarrow T \leftarrow T^6 \leftarrow T^{11} \leftarrow T^7 \leftarrow T \leftarrow 0$$

Step 2: The implicit equation is

$$\det M_{(2,0)} = \begin{vmatrix} 0 & 1 & 0 \\ -Y & -Z & 1 \\ X & 0 & -Z \end{vmatrix} = X - YZ.$$

Publications and References

[1] Eliana Duarte and Alexandra Seceleanu. Implicitization of tensor product surfaces via virtual projective resolutions. *Math. Comp.*, 89(326):3023–3056, 2020.

[2] Christine Berkesch, Daniel Erman, and Gregory G Smith. Virtual resolutions for a product of projective spaces. *arXiv preprint arXiv:1703.07631v2*, 2019.

[3] David Cox. Curves, surfaces, and syzygies. In *Topics in algebraic geometry and geometric modeling*, volume 334 of *Contemp. Math.*, pages 131–150. Amer. Math. Soc., Providence, RI, 2003.

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Research Stays ICERM semester in Nonlinear Algebra, Providence, USA October 2018.