The Geometry of Matroids and Gaussoids

MathCoRe Research Project of Xiangying Chen

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Summary

Matroids are an abstraction and a generalization of linear independence. The goal of this project is to investigate matroids in geometric points of view and to extend the theory to gaussoids and related combinatorial structures.

Geometry of Matroids

Simplicial complexes

A *matroid* on a finite set E is a simplicial complex $\mathcal{I} \subseteq 2^E$ such that for any $I_1, I_2 \in \mathcal{I}$ with $|I_1| > |I_2|$ there exists $e \in I_1 \backslash I_2$ such that $I_2 \cup \{e\} \in \mathcal{I}$.

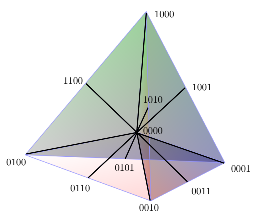
Matroid polytopes

The *matroid base polytope* [1] of a matroid is the convex hull of the indicator vectors of bases.
In the representable case, it is the image of the closure of a torus orbit in the Grassmannian under the moment map.

Theorem ([1]). *A polytope $\mathcal{P}$ is the base polytope of some matroid iff its vertices are 0/1-vectors and its edges are translation of* $\mathbf{e}_i - \mathbf{e}_j$.

Bergman fans

The *Bergman fan* [2] Σ_M of a matroid M is the fan over the order complex of nonempty proper flats.
In the representable case, it is a simplicial fan structure on the tropicalization of the corresponding hyperplane arrangement complement.
The log-concavity of characteristic polynomials of matroids is proven by a Hodge theory of Bergman fans. [3]



The Bergman fan of $U_{3,4}$

New tools to understand matroids: Stanley-Reisner rings, theory of lattice polytopes, intersection theory, etc.

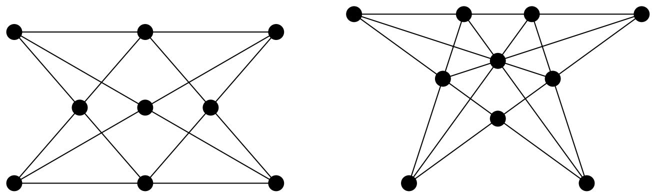
Relation to Complexity Reduction

Independence is an important property in mathematics. Investigations on the geometry of independence can uncover symmetries and new connections of these structures which lead to complexity reduction.

Representations of Matroids

A matroid M is *representable* over some field $\mathbb{K}$ if there is a family of vectors in some vector space over $\mathbb{K}$ whose linear independence is the same as that of M.

- vector/point configurations, hyperplane arrangements
- parameter space: Grassmannian
- universality of realization space



Left above: the non-Pappus matroid, not representable over any field, but representable over some division rings.
Right above: the Perles configuration, not representable over $\mathbb{Q}$, but representable over $\mathbb{Q}(\sqrt{5})$.

Gaussoids and Coxeter Matroids

Gaussoids encode conditional independence among Gaussian distributed random variables.
Let $\mathcal{A}$ be the set of symbols $(ij|K)$ where $i, j \in E$ distinct and $K \subseteq E\backslash\{i,j\}$. $\mathcal{G} \subseteq \mathcal{A}$ is called a *gaussoid* [4] if for all $i, j, k \in E$ distinct and $L\backslash\{i,j,k\}$:

1. $\{(ij|L), (ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ik|L), (ij|kL)\} \subseteq \mathcal{G}$,
2. $\{(ij|kL), (ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ij|L), (ik|L)\} \subseteq \mathcal{G}$,
3. $\{(ij|L), (ik|L)\} \subseteq \mathcal{G} \Rightarrow \{(ij|kL), (ik|jL)\} \subseteq \mathcal{G}$,
4. $\{(ij|L), (ij|kL)\} \subseteq \mathcal{G} \Rightarrow (ik|L)$ or $(jk|L) \in \mathcal{G}$.

A gaussoid is *realizable* if it consists of the vanishing almost-principal minors of a real positive definite symmetric matrix.

Let W be a finite reflection group and P a parabolic subgroup. A *Coxeter matroid* is a subset $M \subseteq W/P$ such that for any $w \in W$ there is a unique maximal element in M with respect to the w-Bruhat order.

The plan is to research into the geometry of these combinatorial structures to get a better understanding on them.

Publications and References

[1] Israel M Gelfand, R Mark Goresky, Robert D MacPherson, and Vera V Serganova. Combinatorial geometries, convex polyhedra, and schubert cells. *Advances in Mathematics*, 63(3):301–316, 1987.

[2] Federico Ardila and Caroline J Klivans. The bergman complex of a matroid and phylogenetic trees. *Journal of Combinatorial Theory, Series B*, 96(1):38–49, 2006.

[3] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. *Annals of Mathematics*, 188(2):381–452, 2018.

[4] Tobias Boege, Alessio D'Alì, Thomas Kahle, and Bernd Sturmfels. The geometry of gaussoids. *Foundations of Computational Mathematics*, 19(4):775–812, 2019.

Cooperations Currently I am working on a project with Thomas Kahle and Tobias Boege.