Geometry and Inference of Gaussian conditional independence

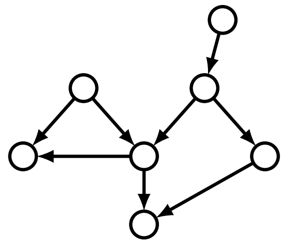
MathCoRe Research Project of Tobias Boege

Funded Fellow since October 2018 — Advisors: Thomas Kahle, Volker Kaibel — Mentor: Eliana Duarte

Otto von Guericke Universität Magdeburg · DFG-Graduiertenkolleg Mathematische Komplexitätsreduktion · Max-Planck-Institut für Dynamik komplexer technischer Systeme Magdeburg

Summary

Conditional independence (CI) is a fundamental relation between random variables in statistics and a common type of constraint for statistical models. The structure of the sets of true CI statements is rich even when one restricts to regular multivariate Gaussian distributions. This class is wide enough to include popular graphical models such as Bayesian networks.
Gaussian CI constraints can be understood as algebraic subsets of the cone of positive-definite (covariance) matrices. Its ties to real geometry imply computability results whose analogues have not yet been established for CI structures of discrete random variables. Within these natural confines, certain aspects of Gaussian CI can become arbitrarily complex. The goal of this project is to understand this inherent complexity in Gaussian CI and investigate more tractable approximations to it.



Results

- The CI implication problem for Gaussians is decidable: $\bigwedge \mathcal{L} \Rightarrow \bigvee \mathcal{M}$ holds if and only if a polynomial defined by $\mathcal{M}$ vanishes on a real variety defined by $\mathcal{L}$.
- As n grows, there are arbitrarily long valid inference rules for n-variate Gaussians, which are logically independent of all lower-dimensional valid inference rules.
- The valid inference rules in the smallest non-trivial case $n = 3$ are logically complete for all inference rules with $|\mathcal{L}| \le 2$. These are the *gaussoid* axioms.
- Gaussoids can be found such that the algebraic degree over $\mathbb{Q}$ of *any* covariance matrix realization can be arbitrarily high.

Irreducible complexity & Approximations

The results suggest that inference properties of Gaussian CI are inherently hard, but not impossible, to describe in a sound and complete manner.

Its properties can be discovered peu à peu and *imitating* its fundamental geometric features combinatorially leads to various interesting approximations of Gaussian CI:

- Gaussoids, oriented gaussoids,
- Semimatroids and
- Algebraic Gaussians.

These approximations satisfy (complementary) subsets of the regular Gaussian CI inference properties. They are incomplete but can be treated more efficiently, using:

- SAT solvers,
- Linear programming and
- Gröbner bases, respectively.

The geometry of Gaussian CI

A conditional independence statement $(ij|K)$ requires that variables i and j be (stochastically) independent when the set of variables K is controlled. For Gaussians, $(ij|K)$ is equivalent to an algebraic equation on the positive-definite covariance matrix Σ:

$$(ij|K) \;\Leftrightarrow\; \det \Sigma_{ij|K} := \det \Sigma_{iK,jK} \overset{!}{=} 0.$$

CI implication

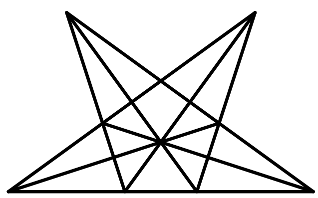
An inference formula $\bigwedge \mathcal{L} \;\Rightarrow\; \bigvee \mathcal{M}$ is *valid* if every Gaussian distribution satisfying the independencies in $\mathcal{L}$ satisfies at least one of those in $\mathcal{M}$. This holds if and only if

$$f_{\mathcal{M}} \text{ vanishes on } \mathrm{PD}_n \cap \mathcal{V}_{\mathcal{L}}, \text{ where}$$
$$f_{\mathcal{M}} = \prod_{(ij|K)\in\mathcal{M}} \det \Sigma_{ij|K} \text{ and}$$
$$\mathcal{V}_{\mathcal{L}} = \{\Sigma \in \mathrm{Sym}_n : \det \Sigma_{ij|K} = 0 \;\forall (ij|K) \in \mathcal{L}\}.$$

Since $\mathbb{R}$ is real-closed, the Positivstellensatz gives an algorithm to decide this.

Encoding of projective incidence geometry

Gaussian CI statements can be used to encode arbitrary incidence relations between points and lines in the real projective plane. Every realization of these statements must also realize the incidence relation.



The Perles configuration requires $\sqrt{5}$ to be realized.

Publications and References

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Cooperations Xiangying Chen, Andreas Kretschmer, Frank Röttger.

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