Lecture 17: Geometric modeling meets algebraic statistics

In this lecture we will enter the word of geometric modeling by defining blending functions that are used to parametrize objects (i.e curves and surfaces) in n-dimensional Euclidean space. We will see that it is it is useful when these functions have one property which is called rational linear precision. Interestingly such property is equivalent to the associated toric variety having ML degree 1.

A Bézier curve of degree n is a function $F : [0, n] \to \mathbb{R}^\ell$ given by

$$F(t) = \frac{1}{n^n} \sum_{i=1}^{n} \binom{n}{i} t^i (n-t)^{n-i} P_i$$

The functions $\binom{n}{i} t^i (n-t)^{n-i}$ for $i \in \{0, \ldots, n\}$ are called blending functions and the Points $P_0, \ldots, P_i$ are called control points. Each $P_i \in \mathbb{R}^\ell$. Note that to each lattice point in the segment $[0, n]$ you associate a blending function.

In $2D$ a Bézier tensor product patch of degree (m, n) is a function $F : [0, m] \times [0, n] \to \mathbb{R}^\ell$

$$F(s,t) = \frac{1}{n^n m^m} \sum_{i=0}^{m} \sum_{j=0}^{m} \binom{m}{i} \binom{n}{j} s^i (m-s)^{m-i} t^j (n-t)^{n-j} P_{(i,j)}$$

The functions $\binom{m}{i}\binom{n}{j} s^i (m-s)^{m-i} t^j (n-t)^{n-j}$for each $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$ are called blending functions and the $P_{(i,j)}$ are called control points.

We can do a similar thing for the dilated simplex $a\Delta_2$, this gives us triangular patches. A toric patch is a generalization of this constriction to an arbitrary domain polytope P.

Notation: P is a lattice polytope. The vectors $n_1, \ldots, n_r$ are the inwatd facing primitive normal vectors of P, $F_1, \ldots, F_r$ are the corresponding facets, $a_1, \ldots, a_r$ are the corresponding integer translates in the facet presentation of P.

$$P = \{p \in \mathbb{R}^d : \langle p, n_i \rangle \geq -a_i, \forall i = 1, 2, \ldots, r.\}$$

The lattice distance function to the face F_i evaluated at $p \in \mathbb{R}^d$ is

$$h_i(p) = \langle p, n_i \rangle + a_i \text{ for } i = 1, \ldots, r$$

We write $h(p) = (h_1(p), \ldots, h_r(p))$. For vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, v^w denotes $\prod_{i=1}^n v_i^{w_i}$. The set $\mathcal{A} = P \cap \mathbb{Z}^d$ denotes the set of lattice points in P, $\mathcal{A} = \{m_1, \ldots, m_n\}$. We will also use a vector of weights for each lattice point $w = (w_1, \ldots, w_n) \in \mathbb{R}^n_{>0}$.

For $j \in \{1, \ldots, n\}$, define $\beta_j : P \to \mathbb{R}$ and $\beta_w : P \to \mathbb{R}$ by

$$\beta_j(p) = \prod_{i=1}^{r} h_i(p)^{h_i(m_j)}, \quad \beta_w(p) = \sum_{j=1}^{n} w_j \beta_j(p).$$

The set of toric blending functions of (P, w) is $\{\beta_{w,j} : j \in \{1, \ldots, n\}\}$ where $\beta_{w,j} : P \to \mathbb{R}$ and $\beta_{w,j}(p) = \frac{w_j \beta_j(p)}{\beta_w(p)}$. Take control points $\{Q_j \in \mathbb{R}^\ell : j \in \{1, \ldots, n\}\}$. The toric patch associated to (P, w) is the map $F : P \to \mathbb{R}^\ell$,

$$F(p) = \frac{1}{\beta_w(p)} \sum_{j=1}^{n} w_j \beta_j(p) Q_j$$

Definition 27. The pair (P, w) has rational linear precision of there is a set of rational functions $\{\hat{\beta}_j : j \in \{1, \ldots, n\}\}$ defined on $\mathbb{C}^d$ satisfying:

1. $\sum_{j=1}^{n} \hat{\beta}_j = 1$
2. The functions $\{\hat{\beta}_j : j \in \{1, \ldots, n\}\}$ define a rational parametrization of $X_{A,w}$, $\hat{\beta} : \mathbb{C}^d \dashrightarrow X_{A,w} \subset \mathbb{P}^{n-1}$. For $t \in \mathbb{C}^d$, $\hat{\beta}(t) = (\hat{\beta}_1(t), \ldots, \hat{\beta}_n(t))$
3. For every $p \in \mathrm{Relint}(P) \subset \mathbb{C}^d$, $\hat{\beta}_j$ is defined and a nonnegative real number.
4. Linear precision:
$$\sum_{j=1}^{n} \hat{\beta}_j(p) m_j = p \text{ for all } p \in P$$

Definition 28. The pair (P, w) has *strict linear precision* if the set of toric blending functions has rational linear precision.

Theorem 34 ([10]). *The pair (P, w) has rational linear precision if and only if $X_{A,w}$ has ML degree one*

To a toric patch we associate the polynomial

$$f = f_{\mathcal{A},w}(t) = w_1 t^{a_1} + w_2 t^{a_2} + \cdots + w_n t^{a_n}$$

and its homogenization

$$F = F_{\mathcal{A},w}(x) = w_1 x^{\hat{a}_1} + w_2 x^{\hat{a}_2} + \cdots + w_n x^{\hat{a}_n}$$

Theorem 35 ([10]). *Let $\mathcal{A} \subset \mathbb{Z}^d$ be such that $\mathbb{Z}\mathcal{A} = \mathbb{Z}^d$, $w \in \mathbb{R}^n_{>0}$ be a vector of weights and*

$$f = f_{\mathcal{A},w}(t) = w_1 t^{a_1} + w_2 t^{a_2} + \cdots + w_n t^{a_n}.$$

Then (P, w) has rational linear precision if and only if the rational function $\psi_{\mathcal{A},w} : \mathbb{C}^d \to \mathbb{C}^d$ defined by

$$t \mapsto \frac{1}{f}\left(t_1 \frac{\partial f}{\partial t_1}, t_2 \frac{\partial f}{\partial t_2}, \ldots, t_d \frac{\partial f}{\partial t_d}\right)$$

is a birational isomorphism.

Theorem 36. *The pair has rationa linear precision if and only if the map $\Phi_F : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ defined by*

$$(x_0 : \cdots : x_d) \mapsto \left(x_0 \frac{\partial F}{\partial x_0} : x_1 \frac{\partial F}{\partial x_1} : \ldots : x_d \frac{\partial F}{\partial x_d}\right)$$

is a birational isomorphism.

New Goal: Classify all complex homogeneous polynomials $F(x_0, \ldots, x_d)$ such that $\Phi_F : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ defines a birational isomorphis. In this case we say F defines a toric polar Cremona transformation (TPCT).

Remark 5. *We say that* $F(x_0,\ldots,x_d)$ *is* homaloidal *if the map*

$$(x_0:\cdots:x_d)\mapsto\left(\frac{\partial F}{\partial x_0}:\frac{\partial F}{\partial x_1}:\ldots:\frac{\partial F}{\partial x_d}\right)$$

defines a birational ismorphism. Homaloidal polynomials appear in a recent characterization of Gaussian statistical models with ML degree one.

Exercise 30. *Prove that F(x,y,z) defines a toric polar Cremona transformation if and only if* $F(ax,by,cz)$ *does for* $a,b,c\in\mathbb{C}^*$. *Prove that F(x,y,z) defines a toric polar Cremona transformation if and only if* $F(x,y,z)^a$ *does for any integer a greater than one.*

We end this lecture with the classification of all polynomials $F(x,y,z)$ that define a toric polar Cremona transformation.

Theorem 37. *A homogeneous polynomial* $F\in\mathbb{C}[x,y,z]$ *that defines a toric polar Cremona transformation is equivalent to one of the following:*

1. $(x+z)^a(y+z)^b, a,b,\geq 1$
2. $(x+z)^a((x+z)^d+yz^{d-1})$ *for* $a\geq 0,\ b,d\geq 1$ *or*
3. $(x^2+y^2+z^2-2(xy+xz+yz))^d,\ d\geq 1.$

The class of polynomials in (1) gives rise to the class of tensor product patches. The polynomial in (2) gives rise to triangular patches when $a=0$ and $d=1$. Otherwise it gives rise to trapezoidal patches. Note that the coefficients of the polynomials give all scalings for with the ML degree drops to one.

Example 61. Consider the polytope $P=\operatorname{conv}((0,0),(1,0),(0,1),(1,1))$. Then

$$P=\{(x_1,x_2)\in\mathbb{R}^2: x_1\geq 0, x_2\geq 0, 1-x_1\geq 0, 1-x_2\geq 0\}$$

The lattice distance functions to the facets of P are the following functions of (x_1,x_2):

$$h_1=x_1,\quad h_2=x_2,\quad h_3=1-x_1,\quad h_4=1-x_2$$

The toric blending function for the point $(0,0)$ is

$$\beta_{\begin{pmatrix}0\\0\end{pmatrix}}(x_1,x_2)=h_1(x_1,x_2)^{h_1(0,0)}h_2(x_1,x_2)^{h_2(0,0)}h_3(x_1,x_2)^{h_3(0,0)}h_4(x_1,x_2)^{h_4(0,0)}=(1-x_1)(1-x_2)$$

$h(0,0)=(0,0,1,1)$.

For this example the polynomials that encode a patch with arbitrary weights are

$$f_{\mathcal{A},w}(s,t)=w_1+w_2s+w_3t+w_4st$$

$$F_{\mathcal{A},w}(x_0,x_1,x_2)=w_1x_0^2+w_2x_1x_0+w_3x_0x_2+w_4x_1x_2.$$

When $w=(1,1,1,1)$ we get $f=(1+s)(1+t)$. The associated map $\psi_{\mathcal{A},w}$ is

$$(s,t)\mapsto\left(\frac{s}{1+s},\frac{t}{1+t}\right).$$

We see that f is the dehomogenization of $(x+z)(y+z)$. If I want to get a parametrization with all the scalings that give me ML degree drop to one, then I compute

$$(ax+cz)(by+cz) = (abxy + acxz + cbzy + c^2).$$

I then dehomogenize by z to get

$$c^2 + acx + cby + abxy$$

So the ML degree one locus of scalings is the image of the map

$$(a, b, c) \mapsto (c^2, ac, cb, ab).$$

If we label the coordinates by w_1, w_2, w_3, w_4, then the image of this map is $V(w_2w_3 - w_1w_4)$